

# Chern and Majorana Modes of Quasicrystals

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(Dated: October 19, 2012)

The topology of quasicrystals is found to have a novel manifestation in the spatial profile of band edge states as topological invariants transform various peaks into doublets of size equals the Chern number. The Chern-dressed peaks form a self-similar pattern encoding topological fingerprints at all length scales. For quasicrystals exhibiting a localization transition, fluctuations about exponentially localized zero modes describe a critical point of the topological phase transition where Majorana modes delocalize. These exotic modes can be captured in their entirety using  $U(1)$  symmetry breaking perturbation where the localization transition is accompanied by the topological transition as edge-localized modes move to the interior, loosing topological protection.

PACS numbers: 71.23.Ft, 05.30.Rt, 42.70.Qs, 73.43.Nq

The revelation that quasicrystals belong to topologically nontrivial phases[1] of matter is an exciting new development that opens new avenues in the frontiers of topological insulators. Quasicrystals (QC) are fascinating ordered structures exhibiting fractal self-similar spectral properties and long range order with regular Bragg diffraction, while having forbidden crystallographic rotational symmetries.[2] Topological insulators are exotic states of matter that are insulators in the bulk but conduct along the edges[3], characterized by topologically protected gapless boundary modes. These edge modes are a manifestation of the nontrivial band structure topology of the bulk[4] and their number equals[5] the topological integer, the Chern number. We will refer to these modes as *edge-Chern* modes.

Key to the topological characterization of Quasicrystals is the translational invariance that shifts the origin of quasiperiodic order[1] and manifests as an additional degree of freedom relating QCs to higher dimensional periodic systems. These shifts of origin known as phasons produce QCs that look locally different but belong to the same isomorphism class[6]. Topological description of QCs require an ensemble of such systems and can be characterized by Chern numbers in view of their mapping to higher dimensions. Explicit demonstration of transport, mediated by the edge mode was shown in a beautiful experiment by pumping light across the QC.[1] Harper and Fibonacci models, the two iconic examples of QCs, are shown[7] to be topologically equivalent and this equivalence is preserved irrespective of whether the quasiperiodic disorder appears in the diagonal or in the off-diagonal term.

This paper elucidates a novel manifestation of topology that is unique to QCs. In 1D QCs, we show that the *band edge modes* encode topological invariants in their spatial profiles. Characterized by goldenmean incommensurability, all Fibonacci peaks (and hierarchy of sub-peaks) of the band edge states of the QCs are transformed into doublets of size equals the Chern number. These Chern-dressed peaks accompanying edge-Chern modes are found in Harper, Fibonacci model as well as in a generalized model that interpolates between the two. In other words, in QCs, the topology introduces a

new length (equals the Chern) that together with competing incommensurate periodicities of the QC provides a hierarchical manifestation of topological invariants.

QCs also provide a new perspective on Majorana modes, zero-energy topologically protected modes at the ends of an infinitely long system with open boundary condition that have been the subject of very intense studies due to possible applications to quantum computing.[8] In QCs where quasiperiodicity induces localization, we show that the fluctuations about exponentially localized zero energy modes describe QCs at onset to a topological phase transition where edge localized modes move to the interior of the chain. To study Majorana away from criticality, we consider a perturbed Harper model with broken  $U(1)$  symmetry system. The system describes a  $p$ -wave superconducting quantum wire and also anisotropic XY spin-1/2 model in a spatially inhomogeneous quasiperiodic magnetic field. These QCs host both the Majorana and Chern modes, two very different type of topologically protected edge modes. Topological phase transition in these quasiperiodic system is accompanied by a localization transition where the zero energy modes move from the edge to the interior of the chain. Edge-Chern modes remain unperturbed by the localization transition.

We consider a 1D chain of spinless fermionic atoms in a lattice described by the Hamiltonian,

$$H(\phi) = - \sum_{\mathbf{s}} c_{\mathbf{s}}^{\dagger} c_{\mathbf{s}+1} + \text{h.c.} - \sum_{\mathbf{s}} V_n(\phi) c_{\mathbf{s}}^{\dagger} c_{\mathbf{s}} \quad (1)$$

Here,  $c_{\mathbf{s}}^{\dagger}$  is the creation operator for a fermion at site  $\mathbf{s}$ . In our studies we have investigated a generalized potential that interpolates between the Harper and the Fibonacci model[7], however, for our presentation here we will restrict to the Harper model with  $V_n = 2\lambda \cos(2\pi(\sigma n + \phi))$ , incommensurate potential characterized by an irrational number  $\sigma$  which we take to be the golden mean  $((\sqrt{5} - 1)/2)$ . Here  $\lambda$  controls the strength of quasiperiodic disorder and  $\phi$  is an arbitrary phase. The Eigenvalue equation, namely the Harper equation,

$$\psi_{n+1}^r + \psi_{n-1}^r + 2\lambda \cos(2\pi(\sigma n + \phi))\psi_n^r = E\psi_n^r \quad (2)$$

exhibits a self-similar spectrum and wave functions at  $\lambda = 1$ [9, 10]. This self-dual point is the critical point for quasiperiodic disorder-induced quantum phase transition from extended ( $\lambda < 1$ ) to exponentially localized phase ( $\lambda > 1$ )[9] with localization length  $\xi = \ln(\lambda)$ .

The incommensurate system is studied by approximating the golden mean,  $\sigma$  by a sequence of rational approximates, the ratio of two consecutive Fibonacci numbers: the Fibonacci sequence is defined by  $F_0 = 0$ ,  $F_1 = F_2 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$ . For any rational approximant  $\sigma = F_{n-1}/F_n$ , the system consists of  $F_n$  bands and  $F_n - 1$  gaps.

In parallel with the well known[4] formalism of quantized Hall conductivity for the 2D bulk problem, one can define an *adiabatic conductivity*  $\sigma_\Phi$  for a 1D ensemble  $H(\phi)$  of chains with periodic boundary condition,

$$\sigma_\Phi = \frac{e^2}{\hbar} c_r = \frac{e^2}{\hbar} \text{Im} \sum_{l=1}^r \int d\phi \left[ \int dk \sum_{n=1}^q \partial_k (\psi_n^l)^* \partial_\phi \psi_n^l \right]. \quad (3)$$

where  $k$  is the Bloch index,  $r$  labels the gap characterized by Chern number  $c_r$  and integration over  $\phi$  corresponds to an ensemble average of set of chains related to each other by translation using phason shifts [6] controlled by  $\phi$ . However, the quantity in the square bracket in Eq. (3) is independent of  $\phi$ , and therefore Chern number can be associated with any  $H(\phi)$ .

For a finite chain, gaps host edge-Chern modes that traverse the gaps as  $\phi$  varies. Variation of energy w.r.t  $\phi$  of these modes illustrates the key concept underlying adiabatic transport[1] as the Chern localized at one of the edges of the chain moves to the other edge after its energy reaches the band. (See inset in Fig. 1) As shown in the Fig. 1, the topological invariants manifest as the number of excursions across the QC as the phase varies adiabatically in its entire range.

We now describe a novel signature of topological invariants that are unique to QCs. This manifestation of topology exists in bulk states that appear at the band-edges, namely the states that borders the gaps and hence reside (energetically) in the immediate vicinity of the tail of the edge-Chern modes. We will refer these modes as *band-Chern modes*. Figure 2 displays the spatial profiles of topologically trivial (ground state) and the band-Chern 4 modes. We recall that all previous scaling analysis of the wave functions for QCs have been carried out for special points of the spectrum such as mid-band points[10] or the band edge points corresponding to maximum or minimum energy states. Such states are topologically trivial. These studies describe incommensurate states as consisting of a central or main peak and a sequence of sub peaks at Fibonacci distance from the central peak as shown in Fig. 2. The subpeaks have intensity that approach a well defined universal ratio  $\zeta$  describing the ratio of the subpeaks to the central peak, for peaks far from the central peak. The wave functions display self-similarity as the structure around subpeaks approach a scaled version of the structure around the central peak.

A comparison of spatial profiles of band-Chern modes with topologically trivial states ( Figs 2, 3) shows that Fibonacci

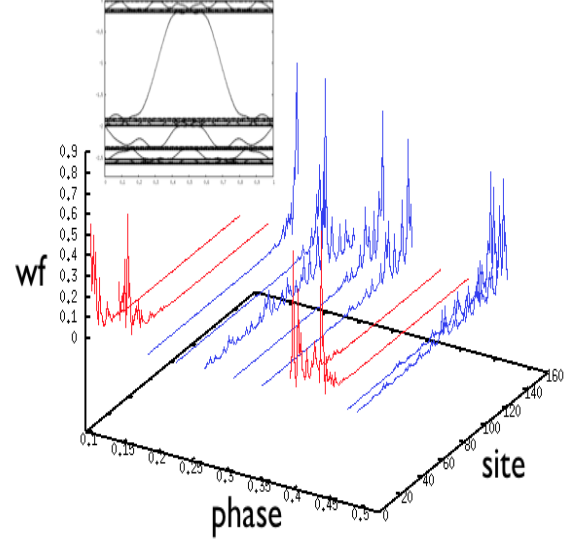


FIG. 1: (color online) Figure illustrates the adiabatic conductivity for the Chern 3 mode in the Harper model for  $\lambda = 1$ . Edge state makes three distinct excursion's between the two edges ( shown with red and blue) of the chain as the phase  $\phi$  varies adiabatically. The small inset at the top shows part of the spectrum with edge-Chern modes corresponding to Cherns 1, 2 and 3 in the bulk gaps.

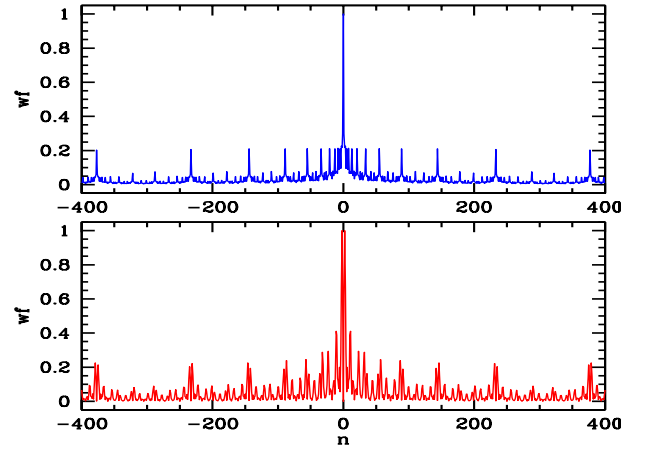


FIG. 2: (color online) Spatial profile of the ground state (top) and band-Chern state with Chern number 4 ( bottom). Topology splits the Fibonacci peaks into a doublet, preserving a self-similar pattern.

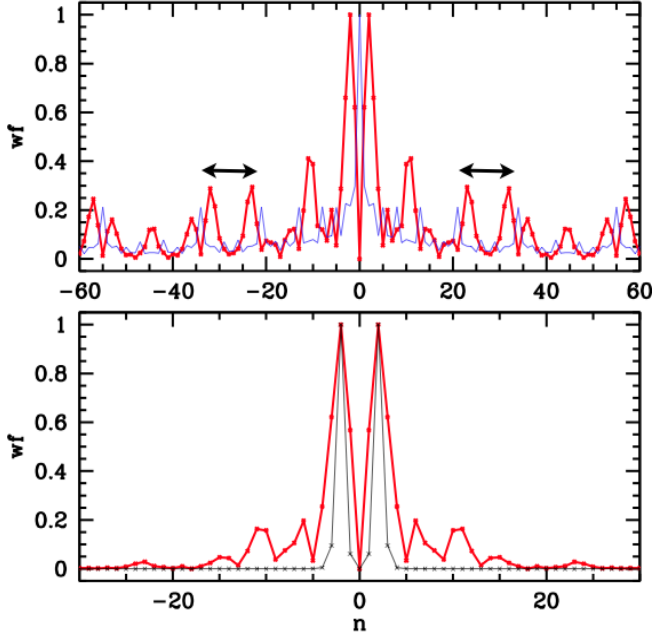


FIG. 3: (color online) The top part shows the blowup of the central peak for ground (blue) and band-Chern 4 (red) for  $\lambda = 1$ . The asymmetry seen in the doublet is found to be restored asymptotically. The black double arrow-line shows the unusual peak separation of 9 in Fibonacci landscape. The bottom shows the corresponding plots for  $\lambda = 1.1$  (red) and  $\lambda = 10$  (black) illustrating topological encoding irrespective of the amount of disorder.

peaks split into two-peak structure with spacing equals the Chern number. This dressing of the Fibonacci peaks with the Cherns was found to be true for all band edge states suggesting that the topology manifests as a new length in quasicrystalline systems. It is quite intriguing that a new length in the Fibonacci landscape preserves the self-similar fractal pattern that characterizes quasiperiodic systems as structures around subpeaks approach a scaled version (with a universal scaling ratio) of the peaks around the central peak.

Fig. 3 compares the spatial profile near the main peak for topologically trivial and non-trivial states. One of the interesting outcomes of the self-similar fractal pattern with Chern-dressed Fibonacci peaks is the existence of certain non-Fibonacci peak separations such as 9 that are unrelated to the Cherns. Common myth that peak separations in goldenmean QCs must be a Fibonacci is not valid for topological states. Such anomalous peaks follow the period-3 pattern[10] intertwined with the self-similar pattern. Simple consideration that pattern inbetween two Fibonacci should be symmetrical about the center accounts for such peak separations.

As illustrated in Fig. 3, the self-similar spatial profiles decay exponentially for  $\lambda > 1$ . For infinite disorder, these states evolve into a dimer of size equals the Chern number, referred as *Chern-dimer mapping*. [13] Therefore, in QCs exhibiting localization transition, the spitting of the peaks into doublets is a natural consequence of the fact that in the localized phase

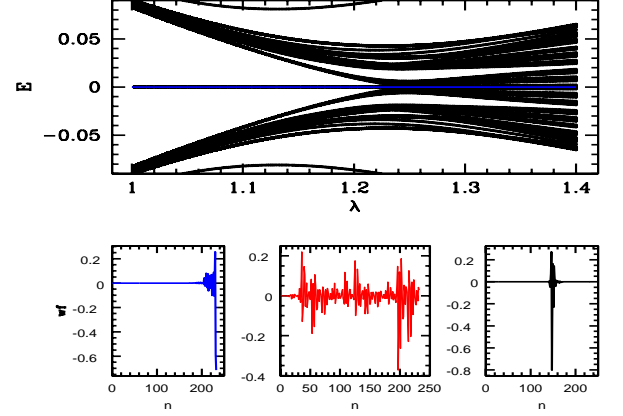


FIG. 4: (color online) (Top) For a finite chain with  $\Delta = .2$ , figure shows the energy spectrum near  $E = 0$  showing the Majorana mode for  $\lambda < 1.2$  (in blue) and Chern-1 mode in the upper gap. (Bottom from left to right) respectively shows the wave functions of  $E = 0$  mode below, at and above the topological quantum phase transition. The edge localized zero energy mode delocalizes at the onset to transition and moves to the center of the chain in topologically trivial phase.

with point-spectrum, band edge modes are the only surviving part of the band structure and therefore must encode topology. The Chern-dimer mapping was demonstrated explicitly in the large  $\lambda$  limit as the system maps to a two-level system. [13, 14] For  $\lambda \rightarrow \infty$ , the mapping is also valid for rational  $\sigma$ . However, for the incommensurate case, the mapping persists for finite  $\lambda > 1$  and transforms into global mapping at  $\lambda = 1$  existing at all length scales. Chern dimer mapping along with self-similar structure of the ground state suggests that spatial profile of the band-Chern mode can be thought of as a convolution of a non-topological state and the  $\lambda \rightarrow \infty$  state.

We will now discuss QCs that in addition to Chern modes also support Majorana modes.[8] We first note that in QCs such as the Harper model exhibiting exponential localization, fluctuations about exponentially decaying envelope[11] of zero energy state describes Majorana at the onset to their extinction. From Eq. (2), it follows that the fluctuations  $\eta_n$  about exponentially decaying envelope,  $\psi_n = e^{-n/\xi}\eta_n$  satisfy the following equation,

$$e^{-\xi}\eta_{n+1} + e^{\xi}\eta_{n-1} + 2\lambda \cos(2\pi(\sigma + \phi))\eta_n = E\eta_n \quad (4)$$

with  $e^{\xi} = \lambda$ . For  $E = 0$ , this equation describes a fermionic representation of the zero energy state of a spin-1/2 anisotropic XY-chain with anisotropy  $\Delta = (\lambda - 1)$ [12], in a spatially modulated transverse magnetic field  $V_n$ [11].

The system also describes a  $p$ -wave superconducting quantum wire with spatially inhomogeneous chemical potential  $V_n$  and superconducting gap parameter  $\Delta$ . For all energies, the system is described by the followed coupled set of equations,

$$(1 - \Delta)f_{n+1} + (1 + \Delta)f_{n-1} + V_nf_n = Eg_n \quad (5)$$

$$(1 + \Delta)g_{n+1} + (1 - \Delta)g_{n-1} + V_ng_n = Ef_n \quad (6)$$

Here  $(f_n, g_n)$  represents a two-component particle-hole wave function of the superconducting chain where two components degenerate for  $E = 0$  state. The Hamiltonian corresponding to the system (6) is,

$$H = \sum \frac{1}{2} c_{n+1}^\dagger c_n + \Delta c_{n+1}^\dagger c_n^\dagger + c.c + V_n(c_n^\dagger c_n - 1/2) \quad (7)$$

For  $\Delta \neq 0$ , this perturbed Harper with broken  $U(1)$  symmetry ( $c_j \rightarrow e^{i\theta} c_j$ ) exhibits a gap in the bulk spectrum at  $E = 0$ , a feature common to homogeneous chain with spatially constant  $V_n$ . In finite chains, it supports Majorana modes at  $E = 0$  and the quasiperiodic system also supports Chern modes.

It should be noted that only for  $\Delta = 1 + \lambda$ , the zero energy mode of this system corresponds to the fluctuations in the localized Harper mode. Interestingly, this correspondence point has been shown to be the critical point of the localization transition of the system (6)[11, 12]

As illustrated in the Fig.(4), the quasiperiodic system (6) with open boundary condition is found to support a zero-energy edge localized mode, Majorana modes for  $\lambda < 1 + \Delta$ . These edge localized modes become extended (critical) at  $\lambda = 1$  and localize in the interior of the chain for  $\lambda > 1$ . Therefore,  $\lambda = 1$  describes a topological phase transition that is also accompanied by a spectral transition to localization. Fractal fluctuations of the Harper equation captures the Majorana at its critical point only and complete topological phase with edge localized Majorana requires  $U(1)$  symmetry breaking perturbation. As expected, Majorana remains immune to quasiperiodic disorder and the topological phase transition corresponds to a displacement of the zero energy edge-localized mode to the interior of the chain. This may provide a rather unique system to trace Majorana modes, as the edge modes delocalize at the critical point and then move into the interior of the chain.

The coupled system (Eq. 6) can be written as a single equation with next-nearest neighbor interaction where quasiperiodicity appears simultaneously in both the diagonal and the off-diagonal term. In the delocalized phase, ( $\lambda < 1 + \Delta$ ), the anisotropic model is characterized by a central gap at zero energy, making it topologically distinct from the Harper system that shows no gap at zero energy. Therefore, topological equivalence of QCs with quasiperiodicity in diagonal or off-diagonal term[7] is not valid in this  $U(1)$  symmetry broken system where both the diagonal and off-diagonal terms exhibit inhomogeneity.

Interplay between topology and quasiperiodicity of QCs provide a remarkable example of competing periodicities where Cherns, the aliens adapt to self-similar environment controlled by the Fibonacci. The topology dominates the local length scales, maintaining global Fibonacci landscape. The mathematical framework to describe renormalization studies of QC states with non-trivial topology is an open problem that will be of interest to wide audience interested in fractal characteristics of incommensurate systems.

Quasicrystals described by the Harper equation have been realized in in ultracold atomic gases[15]. Superconducting quantum wires have been shown to be promising candidates for realizing Majorana modes.[16] QCs with broken  $U(1)$  symmetry are unique systems that host both the Majorana and the Chern modes. The Majorana exists in the central gap, the only gap in this multiband system that does not support the Chern modes. The fact that the Majorana and the Chern modes do not coexist (in the same gap) is an intriguing result and its deeper implications and universality remains open. Quasicrystalline superconducting chains or anisotropic spin chains offer a fascinating new set of possibilities to study topological states supporting both the Majorana and the Cherns.

This research is supported by ONR and DGAPA-UNAM IN100310.

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